J Math Chem (2010) 48:558–565
DOI 10.1007/s10910-010-9691-5

ORIGINAL PAPER

Traveling wave patterns in nonlinear reaction–diffusion equations

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Received: 25 June 2009 / Accepted: 26 April 2010 / Published online: 15 May 2010

Abstract In this paper we study a reaction–diffusion model equation with general nonlinear diffusion and arbitrary kinetic orders in the reaction terms, which appears in the applied biochemical modeling. We carry both analytical and numerical studies of the model equation to show the existence of monotone and oscillatory waves. Our numerical computations are illustrated for a particular case of the equation by using different methods which lead to accurate wave profiles and confirm the analytical results.

Keywords Reaction–diffusion equations · Traveling waves · Heteroclinic trajectories · Time-dependent solutions

1 Introduction

Reaction–diffusion equations are of interest because of the many applications in chemistry and biology. The equation that we consider in this paper is of central interest in the framework of applied biochemical modeling for describing many spatial patterns (see, e.g., [1,2]) that has attracted a lot of attention in the recent years. It is a reaction–diffusion equation with general nonlinear diffusion and arbitrary kinetic orders in the reaction terms. The considered model equation, in a scaled form, reads

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right) + u^p - u^q, \ \ x \in R, \ \ t > 0, \tag{1}$$

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for the unknown concentration function $u(x, t)$ depending on the temporal variable t and the spatial variable x, with $m, p, q > 0$. Similar equations to (1) have been studied for the traveling wave problem and, in particular the existence of oscillatory traveling wave solutions has been shown (see [3] and reference therein).

The aim of this paper is to investigate analytically and numerically traveling wave solutions of (1). Especially the reaction terms in (1) is known to cause oscillatory behavior of the solutions.

This paper is organized as follows. In the next section, we analyze traveling wave behavior in (1). This also includes more accurate computation of the traveling wave solutions for (1). In Sect. 3, we numerically study the evolution to traveling waves for (1). Section 4 contains conclusions.

2 Traveling waves

We study the traveling wave behavior in (1), by looking for solutions in the form $u(x, t) = \theta(x - ct) = \theta(z)$, where c is a constant wave speed. Then, (1) is reduced to the following ordinary differential equation (ODE)

$$-c\frac{d\theta}{dz} = \frac{d}{dz}\left(\theta^m \frac{d\theta}{dz}\right) + \theta^p - \theta^q. \tag{2}$$

Our approach to study the traveling wave solutions is to derive and analyze the corresponding phase trajectory equation of the ODE system associated with (2). Therefore, we rewrite (2) as follows

$$-c\theta^{m-1}\frac{d\theta}{dz} = \theta^m \frac{d}{dz}\left(\theta^{m-1}\frac{d\theta}{dz}\right) + \left(\theta^{m-1}\frac{d\theta}{dz}\right)^2 + \theta^{m-1}(\theta^p - \theta^q), \tag{3}$$

and define

$$v = \theta^{m-1}\frac{d\theta}{dz}, \tag{4}$$

to obtain the following ODE system

$$\begin{aligned} \frac{d\theta}{dz} &= \frac{v}{\theta^{m-1}}, \\ \frac{dv}{dz} &= -\frac{(c+v)v + \theta^{m-1}(\theta^p - \theta^q)}{\theta^m}. \end{aligned} \tag{5}$$

We note that both equations in (5) are degenerate for $\theta \to 0$, since we assume $m > 1$ here. In order to resolve the singularity of system (5) at $\theta = 0$, one defines a transformed variable $\tau = \tau(z)$ into (5), as in [4] such that $d\tau/dz = 1/\theta^m(z)$ depending on the solution θ. Then, the nonsingular ODE system of (5) takes the following form

$$\frac{d\theta}{d\tau} = \theta v,$$
$$\frac{dv}{d\tau} = -(c+v)v - \theta^{m-1}(\theta^p - \theta^q). \tag{6}$$

Note that systems (5) and (6) are topologically equivalent in the positive half plane $\{(\theta, v)|\theta > 0, -\infty < v < +\infty\}$. The system (6) is a regular ODE system in R^2 and has the following three stationary points:

$$p_0 = (0,0),\ p_c = (0,-c) \quad \text{and } p_1 = (1,0).$$

Thus, searching for traveling wave solutions of (1) is equivalent to looking for hetereoclinic trajectories of the ODE system (6), which connect the above stationary points. This is equivalent to looking for solutions in graph form $v = v(\theta) > 0, 0 < \theta \leq 1$, fulfilling the ODE

$$\frac{dv}{d\theta} = -\frac{(c+v)v + \theta^{m-1}(\theta^p - \theta^q)}{\theta v}. \tag{7}$$

Now we describe the local behavior of trajectories near the stationary points by using (6) as well as (7). A linearization around $p_0 = (0,0)$, shows that p_0 is a non-hyperbolic point and the eigenvalues of the Jacobian matrix associated with the system (6) at p_0 are 0 and $-c$. The corresponding eigenvectors are $V_1 = (1,0)^T$ and $V_2 = (0,1)^T$, respectively. Because of the nature of this point, we need to use the Center manifold theorem [6] and Taylor expansion of (7) with $p > q$, to obtain an approximate equation of this path locally around p_0. We find that

$$v(\theta) = \frac{\theta^m}{c} + O(\theta^{m+1}). \tag{8}$$

We note that on this center manifold, system (6) takes the form

$$\frac{d\theta}{d\tau} = \frac{\theta^{m+1}}{c} + O(\theta^{m+2}),$$
$$\frac{dv}{d\tau} = \frac{m}{c^2}\theta^{2m-1} + O(\theta^{2m}). \tag{9}$$

Therefore, the system (6) is weakly repelling on the center manifold (8). Thus $p_0 = (0,0)$ is a degenerate unstable node.

A linearization around $p_1 = (1,0)$ shows that p_1 is a stable node if $c \geq 2\sqrt{p-q} = c_*$, and a stable focus if $c < c_*$, while a linearization around $p_c = (0,-c)$ shows that p_c is a saddle point.

These results of linear local analysis show that the only admissible trajectories corresponding to traveling waves are those connecting the points $p_0 = (0,0)$ to $p_1 = (1,0)$. In the next part we show the existence of such trajectories.

2.1 Results for existence of traveling waves

We establish the existence of a heteroclinic trajectory connecting the two stationary points $p0$ and p_1 for $v > 0$, by a similar method in [5]. We show that, for a suitable value of $\mu > 0$, this trajectory does not leave the region

$$B = \{(\theta, v), \theta \geq 0, v \geq 0, v + \mu(\theta - 1) \leq 0\}.$$

This will be so if $f \cdot n < 0$ on the boundary $v + \mu(\theta - 1) = 0$ of the region, where $n = (\mu, 1)$ is the inward normal vector and f is the vector whose components are the right-hand sides of (6). Thus, we have

$$f \cdot n = \mu\theta v - (c + v)v - \theta^{m-1}(\theta^p - \theta^q).$$

Since $v = \mu(1 - \theta)$, we get

$$f \cdot n = \mu(c + 3\mu)\theta - 2\mu^2\theta^2 - \theta^{m-1}(\theta^p - \theta^q) = R(\theta).$$

If $R'(\theta) \geq 0$ in [0, 1], then $R(\theta) \leq 0$ that $R'(\theta) \geq 0$ as follows. We have

$$\begin{aligned} R'(\theta) = \mu(c + 3\mu) - 4\mu^2\theta - (m - 1)\theta^{m-2}(\theta^p - \theta^q) \\ -\theta^{m-1}(p\theta^{p-1} - q\theta^{q-1}) \geq R'(1). \end{aligned}$$

and

$$R'(1) = -\mu^2 + c\mu - (p - q) = 0,$$

if $\mu = c/2 + \sqrt{(c/2)^2 - (p - q)}$. This implies that the unstable manifold of p_0 enters the region B and joins p_1 to form a heteroclinic trajectory. This demonstrates the existence of heteroclinic trajectories and hence the existence of traveling waves. This will numerically be shown in the next subsection.

2.2 Numerical approximations

To find heteroclinic trajectories which correspond to traveling waves, we solve numerically the phase trajectory Eq. (7) for increasing u. Figures 1 and 2 show the numerical solution of (7) for $m = p = 2$ and $q = 1$ by using an adaptive step Runge–Kutta scheme of fourth order [7] and initial conditions have been estimated from (8). In Fig. 1, we display a sketch of the (θ, v) phase plane containing a solution trajectory connecting the stationary points $p_0 = (0, 0)$ and $p_1 = (1, 0)$, for wave speed $c = 1.4$ with local forms as given by (8). In Fig. 2, we show a sketch of the (θ, v) phase plane containing a oscillatory trajectory connecting the stationary points $p_0 = (0, 0)$ and $p_1 = (1, 0)$, for speed $c = 0.5$ with local form as given by (8).

To show approximations of such waves corresponding to these trajectories, we solve the ODE system (6) by using the fourth order adaptive step Runge–Kutta scheme.

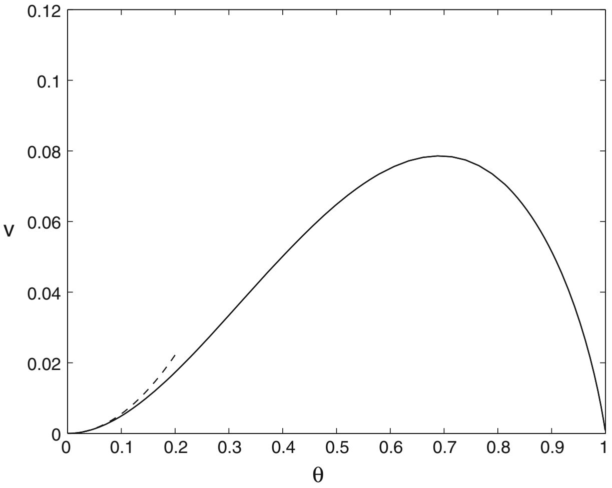

Fig. 1 Sketch showing a trajectory connecting p_0 to p_1 in the phase plane (θ, v) with the local form (*dashed line*) given by (8)

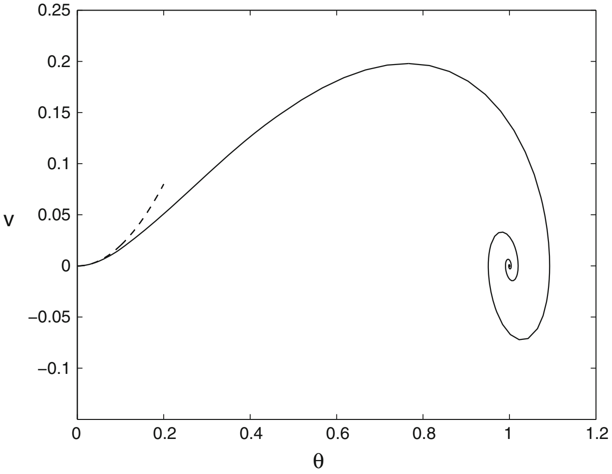

Fig. 2 This figure shows a oscillatory trajectory connecting p_0 to p_1 in the phase plane θ, v with the local form (*dashed line*) given by (8)

In Fig. 3 we show the numerical solution for the speed $c = 1.4$ while in Fig. 4 for the speed $c = 0.5$.

3 Time-dependent solutions: evolution for traveling waves

In this section we study the development time for the traveling wave and its shape by solving an initial-boundary-value problem for the time-dependent partial differential Eq. (1) where the traveling wave emerges as the long time solution.

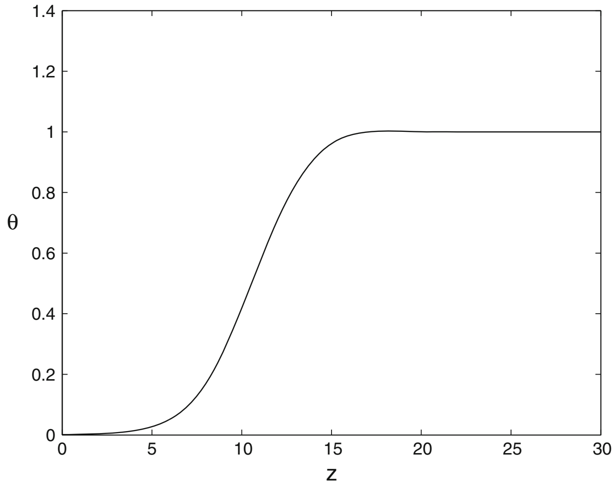

Fig. 3 This figure shows a traveling wave solution corresponding to the trajectory shown in Fig. 1

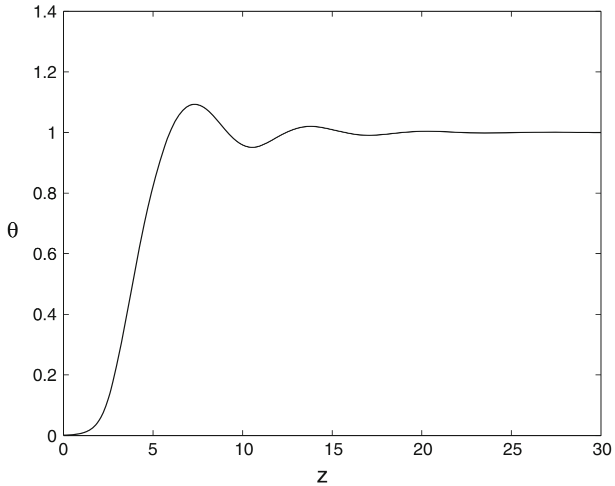

Fig. 4 This figure shows a oscillatory traveling wave solution corresponding to the trajectory shown in Fig. 2

We solve the initial-boundary-value problem consisting of (1) with $m = p = 2$ and $q = 1$ for appropriate boundary and initial conditions, by discretizing the space derivatives and integrating the resulting ordinary differential equations in time along constant $x = x_i$ lines as follows. Letting the space step is dx, the differencing yields

$$\frac{du_i}{dt} = \frac{J_i - J_{i-1}}{dx} + u_i(u_i - 1), i = 1, 2, \ldots, N, \tag{10}$$

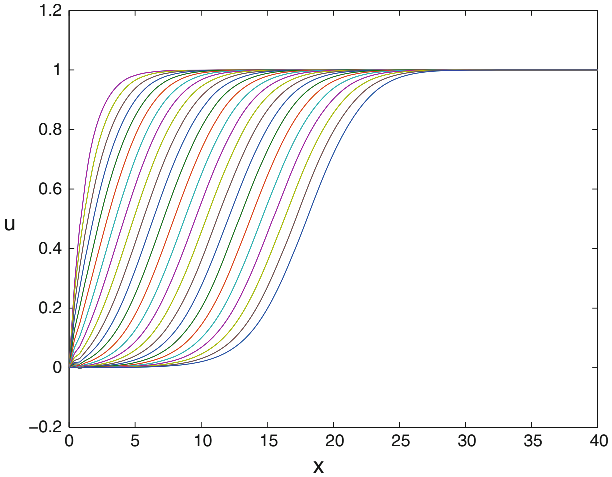

Fig. 5 Sketch showing the solution of (10) with (11) for initial conditions (12) with $\alpha = 0.8$, where we have plotted u as function of space x at equal intervals of time. This shows the evolution to a stable traveling wave

where

$$J_i = \left(\frac{u_{i+1} + u_i}{2}\right)^2 \left(\frac{u_{i+1} - u_i}{dx}\right), i = 0, 1, \ldots N. \tag{11}$$

Initial conditions which are in the form

$$u(x, 0) = 1 - \exp(-\alpha x), \alpha > 0, \tag{12}$$

with appropriate boundary conditions gave rise to traveling waves whose speed decreased as the rate of α of the initial conditions increased. In Figs. 5 and 6 we present two examples of the numerical solutions of (10) with (11) by using the adaptive step Runge–Kutta scheme of fourth order and different choices of initial conditions (12). Clearly in these figures the numerical solutions show the evolution to traveling waves with different speeds. We note that this form of wave speed dependence of initial conditions is familiar from parabolic partial differential equations [8,9]. Also it is clear that our numerical simulations indicate stability of these traveling waves.

4 Conclusions

In this paper we have studied a reaction–diffusion model equation with general nonlinear diffusion and arbitrary kinetic orders in the reaction terms, which appears in the applied biochemical modeling. We have carried both analytical and numerical studies of the model equation to demonstrate the existence of monotone and oscillatory waves. We have used two different methods for the numerical computation of such waves for particular case of the equation. One of the methods involve the traveling

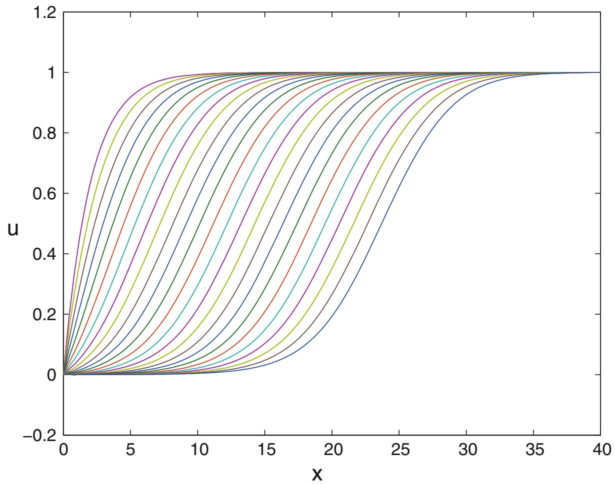

Fig. 6 Plot showing the solution u of (10) with (11) as function of space x at equal intervals of time for initial conditions (12) with $\alpha = 0.5$. Clearly this figure shows a faster wave

wave equations. By using this method, the corresponding trajectories for monotone and oscillatory traveling waves have been approximately determined with high accuracy. The second method is to solve an initial-boundary-value problem for the time-dependent partial differential equation. Starting with appropriate initial conditions, we have numerically approximated these waves. Comparison of the two numerical results obtained as phase plane trajectory and as time-dependent solution, respectively, shows good agreement, in particular the profile of the traveling waves. Moreover, these numerical results confirm the analytical results. Finally, we note that the numerical simulations performed on the partial differential equation problem indicate stability of these traveling waves.

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